TW776: Assignment 1 — For Thursday, 17 February

Assignment Guidelines

- Write a short report to illustrate your work. Use full sentences and include code snippets where applicable.
- Reports are to be handed in during the lecture on the due date.
- Feel free to discuss the work with your classmates, but write your own computer codes, devise your own proofs and examples, and write your own report.

Properties of the Eigenvalues & Vectors of Square Matrices

Use the internet, library or undergraduate class notes to find proofs of the following statements. Rewrite in your own words and cite your source. If you can devise a proof without consulting any source, please say so, there may be one or two bonus marks for this.

After this theoretical investigation, confirm each statement by computing eigenvalues and/or eigenvectors of your favourite 3×3 matrix A (not too trivial). Use the built-in eigenvalue functions in MATLAB (`eig`) or Python (`numpy.linalg.eig`). Hand in copies of your computations.

Unless otherwise stated A is an $n \times n$ real matrix, not necessarily symmetric. Its eigenvalues are λ_1, λ_2, $\ldots$, λ_n, with corresponding eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\ldots$, $\boldsymbol{x}_n$.

(a) $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$. Deduce that A is singular if and only if there exists an eigenvalue equal to 0.

(b) $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. (The trace of a matrix is the sum of its diagonal elements.)

(c) If λ is a nonzero eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.
(Ascher & Greif, problem 2, p. 96)

(d) If A is symmetric, all λ_k are real. (Ascher & Greif, problem 3(a), p. 96)

(e) If A is symmetric and $\lambda_j \neq \lambda_k$, then $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ are orthogonal.

(f) If A is orthogonal, then all its eigenvalues satisfy $|\lambda_k| = 1$.
(Ascher & Greif, problem 11(a), p. 96)

Vector and Matrix Norms

(a) Do Problem 7, p. 97 in Ascher & Greif. That is, confirm that the three properties on p. 79 are satisfied by this definition of norm. In addition, compute $\|\boldsymbol{x}\|_A$, where

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(b) Is the following a well-defined matrix norm according to the four properties on p. 81? Investigate.

$$\|A\| = \max_{1 \le i,j \le n} |a_{ij}|$$

(See Ascher & Greif, p. 96, #(k))