TW776: Assignment 2 **For Tuesday, 1 March**

Assignment Guidelines

- Write a short report to illustrate your work. Use full sentences and include code snippets where applicable.
- Reports are to be handed in during the lecture on the due date.
- Feel free to discuss the work with your classmates, but write your own computer codes, devise your own proofs and examples, and write your own report.
- It is acceptable to consult the library or internet on any problem. But if you do, it is important that you rewrite the material in your own words and cite your source.

Gaussian Elimination

Do Problem 2(b) on p. 144, Ascher & Greif.

Properties of Matrices Arising from the Discretization of PDEs

First read Example 4.17 (p. 91–93) and Example 7.1 (p. 180–184) in Ascher & Greif.

Consider the $N \times N$ second difference matrix

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & & \\ & & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

(a) For the 5×5 case, compute both the LU and LL^T factorizations of A by hand. Generalize your results to the $N \times N$ case.

(b) The eigenvalues of the $N \times N$ version of A are given by

$$\lambda_\ell = 4 \sin^2 \frac{\ell \pi}{2(N+1)}, \qquad \ell = 1, 2, \ldots, N.$$

Confirm this fact for a few values of N, by computing the eigenvalues of A using software, and then comparing the results with this formula. *Hint:* In MATLAB, there is a built-in function for assembling tridiagonal matrices; type `help gallery` and look for `tridiagonal`. In Python, use `scipy.sparse.dia_matrix`.

(c) Based on your results of parts (a)–(b), give two different proofs that A is symmetric positive definite for all $N = 1, 2, \ldots$.

(d) The 2D version of A is defined in Chapter 7, p. 183 in Ascher & Greif, with a formula for the eigenvalues given on the same page. Show that the formula can be written as

$$\lambda_{\ell,m} = 4\Big(\sin^2 \frac{\ell \pi}{2(N+1)} + \sin^2 \frac{m \pi}{2(N+1)}\Big), \qquad 1 \le \ell, m \le N.$$

As in part (b), confirm this formula numerically for a few values of N. *Hint:* In MATLAB type `help gallery` and look for the name of Poisson. In Python, use the provided `sptools` (on the course homepage), or use the external PyAMG package (`pyamg.gallery.poisson`).

(e) Deduce that the 2D version of A is symmetric positive definite as well.

(f) The 2D version of A used above is based on the 5-point stencil shown in Figure 7.2. Now look up the coefficients of the 9-point stencil for the discretization of the Laplacian. Then write down the corresponding matrix A, in the natural ordering. Write a code that assembles this matrix for any N, and experiment with its eigenvalues. Are they all positive? Is this A s.p.d.? Can you find a formula for the eigenvalues, similar to the formula in part (d)?

Bonus Problem (full bonus if you do it on your own, partial bonus if you consult internet or library)

Now derive the formula for the eigenvalues in part (b) for the 1D version of A. *Hint:* Show that $A\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, when written in component form, is

$$-x_{j-1} + 2\,x_j - x_{j+1} = \lambda\,x_j \quad \Longrightarrow \quad x_{j-1} + (\lambda - 2)\,x_j + x_{j+1} = 0, \qquad j = 1, 2, \ldots, N$$

This is a linear second order difference equation, which is solved by putting $x_j = r^j$. Insert into the difference equation, find formulas for r, and enforce the boundary conditions $x_0 = 0$, $x_{N+1} = 0$.