TW776: Assignment 2 (solutions)

Gaussian Elimination

Do Problem 2(b) on p. 144, Ascher & Greif. *Solution found on the internet is appended at the end.*

Properties of Matrices Arising from the Discretization of PDEs

Consider the $N \times N$ second difference matrix

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & & \\ & & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

(a) For the 5×5 case, compute both the LU and LL^T factorizations of A by hand.

$$LU: \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & -\frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -1 \\ 0 & 0 & 0 & 0 & \frac{6}{5} \end{pmatrix}$$

$$LL^T: \quad A = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{4}{3}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{4}} & \sqrt{\frac{5}{4}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{4}{5}} & \sqrt{\frac{6}{5}} \end{pmatrix} \times \text{(transpose)}$$

Generalize your results to the $N \times N$ case. *Trivial.*

(b) The eigenvalues of the $N \times N$ version of A are given by

$$\lambda_\ell = 4 \sin^2 \frac{\ell \pi}{2(N+1)}, \qquad \ell = 1, 2, \ldots, N.$$

Confirm this fact for a few values of N:

```
>> N = 5;
>> A = full(gallery('tridiag',N))

A =

     2    -1     0     0     0
    -1     2    -1     0     0
     0    -1     2    -1     0
     0     0    -1     2    -1
     0     0     0    -1     2
```

```
>> sort(eig(A))

ans =

  2.6795e-001
  1.0000e+000
  2.0000e+000
  3.0000e+000
  3.7321e+000

>> ell = [1:N]';
>> 4*sin(ell*pi/(2*(N+1))).^2

ans =

  2.6795e-001
  1.0000e+000
  2.0000e+000
  3.0000e+000
  3.7321e+000
```

(c) Based on your results of parts (a)–(b), give two different proofs that A is symmetric positive definite for all $N = 1, 2, \ldots$.

Proof 1: A real symmetric matrix A is s.p.d. $\iff$ all $\lambda > 0$. From the formula in part (b) it is clear that all eigenvalues are positive for all $N = 1, 2, \ldots$.

Proof 2: A real symmetric matrix A is s.p.d. $\iff$ A admits a real Cholesky factorization LL^T. We have obtained this factorization explicitly in part (a).

(d) The 2D version of A is defined in Chapter 7, p. 183 in Ascher & Greif, with a formula for the eigenvalues given on the same page. Show that the formula can be written as

$$\lambda_{\ell,m} = 4\Big(\sin^2\frac{\ell\,\pi}{2(N+1)} + \sin^2\frac{m\,\pi}{2(N+1)}\Big), \qquad 1 \le \ell, m \le N.$$

Follows from the trigonometric identity $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$.

As in part (b), confirm this formula numerically for a few values of N:

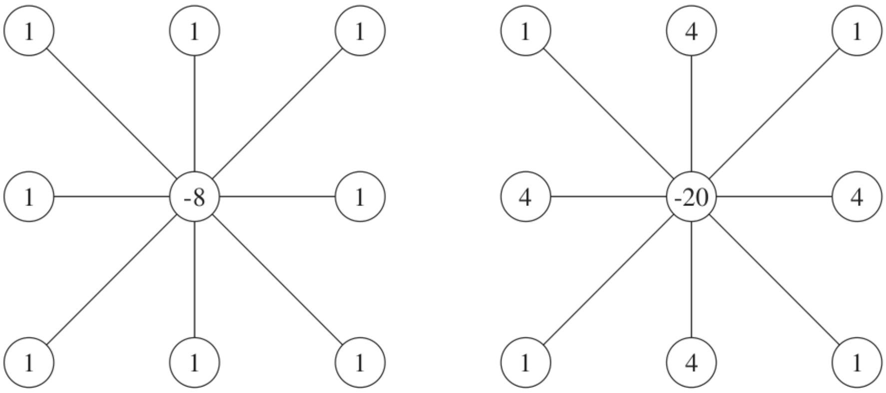
```
>> N = 3;
>> A = full(gallery('poisson',3))

A =

     4    -1     0    -1     0     0     0     0     0
    -1     4    -1     0    -1     0     0     0     0
     0    -1     4     0     0    -1     0     0     0
    -1     0     0     4    -1     0    -1     0     0
     0    -1     0    -1     4    -1     0    -1     0
     0     0    -1     0    -1     4     0     0    -1
     0     0     0    -1     0     0     4    -1     0
     0     0     0     0    -1     0    -1     4    -1
     0     0     0     0     0    -1     0    -1     4

>> sort(eig(A)); reshape(ans,3,3)

ans =

  1.1716e+000  4.0000e+000  5.4142e+000
```

```
  2.5858e+000  4.0000e+000  5.4142e+000
  2.5858e+000  4.0000e+000  6.8284e+000

>> ell = 1:N; em = 1:N; [L,M] = meshgrid(ell,em);

>> 4*(sin(L*pi/(2*(N+1))).^2+sin(M*pi/(2*(N+1))).^2)

ans =

  1.1716e+000  2.5858e+000  4.0000e+000
  2.5858e+000  4.0000e+000  5.4142e+000
  4.0000e+000  5.4142e+000  6.8284e+000
```

(e) Deduce that the 2D version of A is symmetric positive definite as well. *Same arguments as in part (c).*

(f) The 2D version of A used above is based on the 5-point stencil shown in Figure 7.2. Now look up the coefficients of the 9-point stencil for the discretization of the Laplacian.

Apologies, your instructor forgot that there are actually two 9-point stencils, as can be found, e.g., in Saad's book:



Both of these sets of coefficients should be multiplied by -1 (to make s.p.d.) and scaled by factors $1/h^2$ and $1/(6h^2)$, respectively, but we shall just absorb these factors into the right hand side. The stencil on the right is the more accurate of the two (see Saad). For that stencil, the matrix A, in the natural ordering, has dimensions $N^2 \times N^2$ and is block-tridiagonal

$$A = \begin{pmatrix} J & -K & & & \\ -K & J & -K & & \\ & -K & J & & \\ & & & \ddots & -K \\ & & & -K & J \end{pmatrix},$$

where J and K are $N \times N$ tridiagonal matrices given by

$$J = \begin{pmatrix} 20 & -4 & & & \\ -4 & 20 & -4 & & \\ & -4 & 20 & & \\ & & & \ddots & -4 \\ & & & -4 & 20 \end{pmatrix}, \qquad K = \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & & \\ & & & \ddots & 1 \\ & & & 1 & 4 \end{pmatrix}$$

The formula for the eigenvalues of A is somewhat similar to the formula on p. 183 in Ascher & Greif

$$\lambda_{\ell,m} = 20 - 8\big(\cos(\ell\pi h) + \cos(m\pi h)\big) - 4\cos(\ell\pi h)\cos(m\pi h), \qquad 1 \le \ell, m \le N,$$

but not so well-known. Since none of the cosine terms can have the value one or greater, all eigenvalues are positive and A is s.p.d. for all N.

5.2 Cost of Solving a System of Linear Equations Using Gauss-Jordan Elimination

Now we will solve the same 3 x 3 example using Gauss-Jordan Elimination.

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ 1 & 2 & 2 & 2 \\ 4 & 5 & 6 & 7 \end{array}\right] \begin{array}{l} \\ (r2) + (\frac{-1}{3} * r1) \\ (r3) + (\frac{-4}{3} * r1) \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ 0 & \frac{8}{3} & \frac{5}{3} & 1 \\ 0 & \frac{23}{3} & \frac{14}{3} & 3 \end{array}\right] \begin{array}{l} (r1) + (\frac{3}{4} * r2) \\ \\ (r3) + (\frac{-23}{8} * r2) \end{array} \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & \frac{9}{4} & \frac{15}{4} \\ 0 & \frac{8}{3} & \frac{5}{3} & 1 \\ 0 & 0 & \frac{-1}{8} & \frac{1}{8} \end{array}\right] \begin{array}{l} (r1) + (18 * r3) \\ (r2) + (\frac{40}{3} * r3) \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & \frac{8}{3} & 0 & \frac{8}{3} \\ 0 & 0 & \frac{-1}{8} & \frac{1}{8} \end{array}\right] \begin{array}{l} (\frac{1}{3} * r1) \\ (\frac{3}{8} * r2) \\ (-8 * r3) \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To calculate the cost of Gauss-Jordan Elimination, we begin by counting the number of arithmetic operations necessary to convert the coefficient matrix of the original system to a diagonal matrix.

In the first step, we introduced zeros in the first column beginning in row 2. We performed exactly the same operations that we used when we applied Gaussian Elimination to the problem. So the calculations require 4(2) multiplications and 3(2) additions.

For the second step, we needed to eliminate the second element in the third row *and* the second element in the first row. This step required 3 multiplications and 2 additions on each of 2 rows. For the third step, we needed to eliminate the third elements from the first and second rows. This step requires 2 multiplications and 1 addition on each of the 2 rows.

We can write the total operation count so far as

$$4(2) + 3(2) + 2(2) = 18 \quad \text{multiplications and}$$

$$3(2) + 2(2) + 1(2) = 12 \quad \text{additions}$$

Once the matrix was in diagonal form, we divided the diagonal element and the right-hand-side element of each row by the that row's diagonal element. This last step required 3 multiplications. Hence the total operation count for Gauss-Jordan on our 3 x 3 example is 21 multiplications and 12 additions. (Recall that Gaussian Elimination required 17 multiplications and 11 additions.)

Now let us consider solving a general *n* x *n* system using Gauss-Jordan Elimination.

Step 1: Introduce zeros in the first column.

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} & b'_2 \\ 0 & a'_{32} & a'_{33} & \dots & a'_{3n} & b'_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & \dots & a'_{nn} & b'_n \end{array}\right]$$

In the general case, there are $n-1$ rows on which to perform $n+1$ multiplications and n additions.

Step 2: Introduce zeros in the second column.

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & a'_{13} & \dots & a'_{1n} & b'_1 \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} & b'_2 \\ 0 & 0 & a''_{33} & \dots & a''_{3n} & b''_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a''_{n3} & \dots & a''_{nn} & b''_n \end{array}\right]$$

In the general case, there are $(n-1)$ rows on which to perform (n) multiplications and $(n-1)$ additions.

We continue eliminating elements below and above the diagonals for columns three through *n*-1. The last step is to introduce zeros above the diagonal element in the the n^{th} column of the matrix. Once this step is complete we have a diagonal matrix.

Step n: Introduce zeros in the n^{th} column.

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & \dots & 0 & b'''_1 \\ 0 & a'_{22} & 0 & \dots & 0 & b'''_2 \\ 0 & 0 & a''_{33} & \dots & 0 & b'''_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a'''_{nn} & b'''_n \end{array}\right]$$

In the general case, there are $(n-1)$ rows on which to perform 3 multiplications and 2 additions.

Once we have reduced the first *n* columns of the augmented matrix to a diagonal matrix, we must divide each row by its diagonal element. This step requires *n* multiplications.

At this point, column (*n*+1) of the augmented matrix is the solution of the system of equations. Adding together all of the operations required for Gauss-Jordan Elimination for a general system of *n* linear equations, we find that there are

$(n - 1)(n + 1) + (n-1)(n) + \dots + (n -1)(3) + n$ multiplications

$(n -1)(n) + (n - 1)(n - 1) + \dots + (n -1)(2)$ additions

As before, we can use the expression for the sum of the first *n* integers to rewrite the number of multiplications required for Gauss-Jordan Elimination as

$$n + (n-1)\{(n+1) + n + \ldots + 3\} \;=\; n + (n-1)\sum_{i=3}^{n+1} i$$
$$= \; n + (n-1)\left\{\sum_{i=1}^{n} i \; - 1 - 2 + (n+1)\right\} \;=\; n + (n-1)\left\{\frac{n(n+1)}{2} + n - 2\right\}$$
$$= \; \frac{n^3}{2} + n^2 - \frac{5n}{2} + 2.$$

The number of additions required for Gauss-Jordan Elimination is

$$(n-1)\{n + (n-1) + \ldots + 2\} \;=\; (n-1)\sum_{i=2}^{n} i \;=\; (n-1)\left\{\sum_{i=1}^{n} i \; - 1\right\}$$
$$= \; (n-1)\left(\frac{n(n+1)}{2} - 1\right) \;=\; \frac{n^3}{2} - \frac{3n}{2} + 1.$$

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