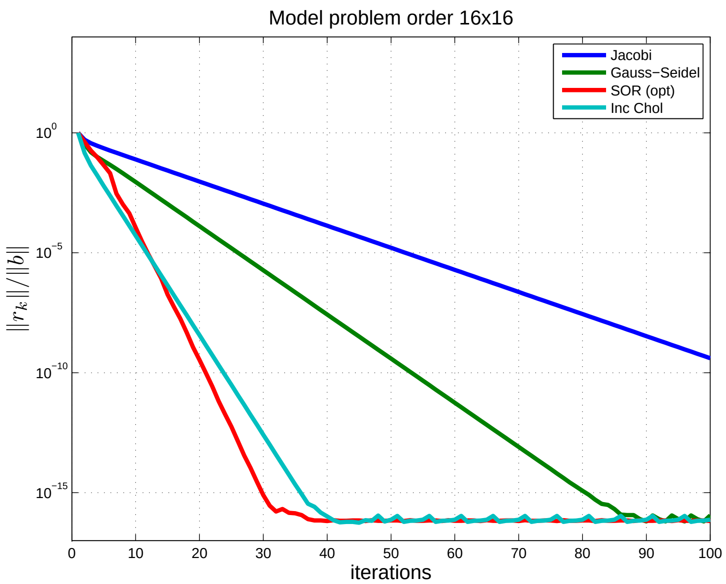
TW776: Assignment 5 — For Thursday, 21 April

Assignment Guidelines

- Write a short report to illustrate your work. Use full sentences and include code snippets where applicable.
- Reports are to be handed in during the lecture on the due date.
- Feel free to discuss the work with your classmates, but write your own computer codes, generate your own figures, devise your own proofs and examples, and write your own report.
- It is acceptable to consult the library or internet on any problem. But if you do, it is important that you rewrite the material in your own words and cite your source.
- Unless told otherwise, use built-in functions in MATLAB or Python as far as possible. Do not write your own. This is not a course in programming.

Stationary Iterative Methods

Consider the convergence curves of the Jacobi, Gauss-Seidel, SOR and Incomplete Cholesky methods for the $N^2 \times N^2$ model problem, as given in the handout of April 6:



(a) Implement the four iterations in your software system to reproduce the figure. For the right hand side, use a random vector $\boldsymbol{b}$. Hints: (a) The optimal ω for SOR is given on p. 190 in Ascher & Greif. (b) The incomplete Cholesky factorisation can be computed using `cholinc` in MATLAB, and `sptools.cholinc` in Python.

(b) Increase the size of the matrix to say 400×400 and generate a similar figure. Discuss and explain the main differences between the two figures. Which method wins for larger matrices?

(c) Generate a table of the spectral radii $\rho(T)$ of the four iteration matrices $T = I - M^{-1}A$, for increasing values of N. [1] By analyzing the data, give your best estimate for the value of p in $1 - \rho(T) \sim c/N^p$ for each of the four methods. Are these values consistent with the theory as summarized in Example 7.8, p. 193? (Incomplete Cholesky excluded.)

[1] Here it is ok to compute the inverse but only for academic purposes!

The Optimal SOR Parameter I

The optimal ω for the SOR iteration can be obtained theoretically in the case of the model problem; it is in fact given on p. 190 in Ascher & Greif. Here we check this value numerically.

Consider the SOR preconditioner M, defined at the top of p. 190. For $N = 15$, compute and plot $\rho(T) = \rho(I - M^{-1}A)$ as a function of ω on $[0, 2]$. Mark the location of the local minimum of the graph as well as the value of the optimal ω as defined in Example 7.7, p. 190. How well does theory agree with experiment? Repeat for at least one other value of N.

The Optimal SOR Parameter II

Here we investigate the optimal SOR parameter by (mostly) a hand calculation. For

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

show that the SOR iteration matrix is

$$T = \begin{pmatrix} 1-\omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1-\omega) & 1-\omega+\frac{1}{4}\omega^2 \end{pmatrix}$$

Show further that if $\omega > 0$ the eigenvalues of T are

$$\lambda_{1,2} = 1 - \omega + \frac{1}{8}\omega^2 \pm \frac{\omega}{8}\sqrt{16 - 16\,\omega + \omega^2}$$

Use software to plot $|\lambda_1|$ and $|\lambda_2|$ as well as $\rho(T)$ as a function of ω on $[0, 2]$. Deduce that the optimal value of ω for this problem is $8 - 4\sqrt{3} = 1.07\ldots$.

A new preconditioner?

Jacobi uses the diagonal part of A as preconditioner. What if we define a new preconditioner M by the *tridiagonal* part of A? Remember that a good preconditioner should have two properties: (i) it should be faster to solve $M\boldsymbol{p} = \boldsymbol{r}$ than it is to solve the original system, and (ii) we need $M \approx A$ so that convergence is fast. Discuss this new preconditioner with reference to (i) and (ii). To decide (ii), investigate the speed of convergence by actually implementing the iteration to solve the model problem. How does this method compare to Jacobi and Gauss-Seidel in terms of speed of convergence?