Probleem 1: (a) $\|A\|_1 = 2$, (b) $\|A\|_\infty = 3$

(c) $A^TA = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$; $\begin{vmatrix} 4-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (4-\lambda)(2-\lambda) - 4 = 0$$
$$8 - 6\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 6\lambda + 4 = 0$$
$$\lambda = \frac{6 \pm \sqrt{36-16}}{2} = 3 \pm \sqrt{5}$$
$$\|A\|_2 = \sqrt{3+\sqrt{5}} \doteq 2.29 \longrightarrow$$

Probleem 2: (a) Waar / True $(QQ^T = I \Rightarrow Q^{-1} = Q^T)$

(b) Waar / true (alle $\lambda > 0 \Rightarrow \det(A) \neq 0$)

(c) Waar / true $(\kappa_2(Q) = 1)$

(d) Vals / false (Hilbert matrix is counterexample)

Probleem 3:

(a) Full $\underset{\sim}{LL^T}$ (cholesky) $\Rightarrow$ flop $\sim \frac{1}{3}n^3$, $p = 3$

(b) Tridiag LU $\Rightarrow$ flops $\sim 8n$, $p = 1$

(c) Forward substitution $\Rightarrow$ flops $\sim n^2$, $p = 2$

(d) Full inverse $\Rightarrow$ flops $\sim \frac{8}{3}n^3$ (2), $p = 3$

(e) Use $A = LU \Rightarrow \det(A) = \det(L)\det(U)$
so same cost as LU $\Rightarrow$ $p = 3$.

Probleem 4:

(a) Flop count is higher (compare (a) and (d) above)

(b) Destroys zeros more so than LU, for sparse A.

Probleem 5:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} l_{11} & & \\ l_{12} & l_{22} & \\ l_{13} & l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ & l_{22} & l_{21} \\ & & l_{22} \end{pmatrix}$$

Kolom 1:

$$1 = l_{11}^2 \Rightarrow l_{11} = 1 \quad \text{(pos diagonal)}$$

$$1 = l_{12} l_{11} \Rightarrow l_{12} = 1$$

$$0 = l_{13} l_{11} \Rightarrow l_{13} = 0$$

Kolom 2:

$$a = l_{12}^2 + l_{22}^2 \Rightarrow l_{22} = \sqrt{a-1} \quad (a > 1)$$

$$1 = \overset{=0}{l_{13}} l_{12} + l_{21} l_{22} \Rightarrow l_{21} = \frac{1}{\sqrt{a-1}}$$

Kolom 3:

$$1 = \underset{=0}{l_{13}^2} + \underbrace{l_{21}^2}_{\frac{1}{\sqrt{a-1}}} + l_{22}^2$$

$$l_{22}^2 = 1 - \frac{1}{a-1} = \frac{a-2}{a-1}$$

$$l_{22} = \sqrt{\frac{a-2}{a-1}} \quad (a > 2)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \sqrt{a-1} & 0 \\ 0 & \frac{1}{\sqrt{a-1}} & \sqrt{\frac{a-2}{a-1}} \end{pmatrix} \qquad \underline{5}$$

$$A \text{ is s.p.d.} \iff a > 2 \qquad \underline{2}$$

Probleem 6: (a) $\kappa(A) = \|A\| \, \|A^{-1}\|$ $\underline{1}$

(b) Ascher & Greif, p. 142 $\underline{4}$

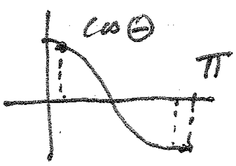
Problem 7: (a) $a \geq 2b$ 2

(b) $\lambda_k = a + 2b \cos\left(\frac{\pi k}{n+1}\right)$

Since $\left|\cos\left(\frac{\pi k}{n+1}\right)\right| < 1$ if $k = 1, 2, \dots n$ 2

$\lambda_k > a - 2b \geq 0$ if $\underbrace{a \geq 2b}_{\text{see (a)}}$.

(c) $a = 2,\ b = 1 \Rightarrow A$ diag. dom $\Rightarrow \lambda_k > 0$



$\max \lambda_k = 2 + 2 \cos \frac{\pi}{n+1}$ $(k=1)$

$\min \lambda_k = 2 + 2 \cos \frac{n\pi}{n+1}$ $(k=n)$

(optional simplification) $= 2 + 2 \cos\left(\frac{n+1-1}{n+1} \cdot \pi\right)$

Use formula pt 6(b) $= 2 - 2 \cos \frac{\pi}{n+1}$

$$K_2(A) = \frac{1 + \cos\frac{\pi}{n+1}}{1 - \cos\frac{\pi}{n+1}}$$ $\frac{4}{4}$

(d) $K_2(A) \sim 10^6 = \frac{\pi^2}{4} n^2 \Rightarrow n = 636,6$

4

For n larger than 637
ill-conditioning can prevent 10 digit accuracy.

Problem 8:

$$A_1 \quad \left[\begin{array}{ccccc|c} \times & \times & \times & \times & \times & \times \\ \times & \times & & & & \times \\ \times & & \times & & & \\ \times & & & \times & & \\ \times & & & & \times & \times \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \times & \times & \times & \times & \times & X \\ 0 & X & F & F & F & X \\ 0 & F & X & F & F & X \\ 0 & F & F & X & F & X \\ 0 & F & F & F & X & X \end{array}\right]$$

After 1 step fill-in produced dense matrix, so plus backsubs, which is $\sim n^2$.

then it is just regular GE with flops $\sim \frac{2}{3}(n-1)^3 \sim \frac{2}{3}n^3$.

$$A_2: \quad \left[\begin{array}{ccccc|c} \times & & & & * & \times \\ & \times & & & \times & \times \\ & & \times & & \times & \times \\ & & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} \times & & & & \times & \times \\ & \times & & & \times & \times \\ & & \times & & \times & \times \\ & & & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \end{array}\right]$$

Step 1: $3 \ */\div$ and $2 \ +/-$ $= 5$ flops

Each subsequent step involves the same amount of work (with the exception of the last). Therefore, the total flop count for the reduction to upper-triangular form is $\sim 5(n-1) \sim 5n$, $n \gg 1$.

Back-substitution:

$$\left[\begin{array}{ccccc|c} \times & & & & \times & \times \\ & \times & & & \times & \times \\ & & \times & & \times & \times \\ & & & \times & \times & \times \\ & & & & \times & \times \end{array}\right]$$

Step 1: $1 \ */\div$

Steps $2-n$: $2 \ */\div$ and $1 \ +/-$ $= 3$ flops

Total count for back substit $\sim 3(n-1)+1 \sim 3n$, $n \gg 1$.

Total cost for solving $A_2 \underline{x} = \underline{b}$ is $\sim 8n$, $n \gg 1$.

(b) $A_1 \to O(n^3)$ vs $A_2 \to O(n) \Rightarrow A_2$ wins.

Approximate minimum degree ordering.

<u>Problem 9</u>: Ascher & Greif, p. 138 (eq (5.1)) shows that a small $\|\underline{r}\|$ does not guarantee a small $\|\underline{x} - \tilde{\underline{x}}\|$ when $\kappa(A)$ is large.