

Diagonal Ramsey Numbers in Multipartite Graphs

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Abstract

The notion of a graph theoretic Ramsey number is generalised by assuming that both the original graph whose edges are arbitrarily bi-coloured and the sought after monochromatic subgraphs are complete, balanced, multipartite graphs, instead of complete graphs as in the classical definition. Some small multipartite Ramsey numbers are found, while upper and lower bounds are established for others. Analytic arguments as well as computer searches are employed.

Keywords: set/size multipartite Ramsey number, multipartite/circulant graph.

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1 Introduction

The classical graph theoretic Ramsey number $r(m, n)$ may be defined as the smallest natural number p such that, if the edges of the complete graph K_p are arbitrarily coloured using the colours red and blue, then either a red K_m or a blue K_n will be forced as subgraph.

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Since Ramsey [32] introduced the problem of finding similar numbers within the realm of formal logic in 1930, the field of Ramsey theory has grown considerably and has branched out to many sub-disciplines of mathematics. There have also been many generalisations within graph theory. One possible graph theoretic generalisation is obtained when defining the Ramsey number $r(F, H)$ as the least natural number p such that, if the edges of K_p are coloured arbitrarily using the colours red and blue, then a red F or a blue H (or both) is forced as subgraph [6]. This Ramsey problem has been (partially) solved for various specific classes of the graphs F and H , such as paths, trees, stars, cycles, books, or combinations of these (see, for example, [4], [5], [18] and [31]). Other generalisations of the classical definition involve using more than two colours when colouring the edges of K_p (see, for example, [13], [24], [25], [26] and [33]) or introducing the concepts of irredundance (see, for example, [2], [3], [7], [8], [10], [11], [12] and [23]) or of upper domination (see, for example, [27] and [28]). There have also been generalisations where the original graph to be coloured is a multipartite graph (see, for example, [13], [15], [16], [20], [24] and [25]).

We too generalise the notion of a Ramsey number by using multipartite graphs in the definition, but with the difference that *both* the original graph whose edges are to be bi-coloured and those which are sought as monochromatic subgraphs are assumed to be multipartite. Using complete, balanced multipartite graphs (instead of complete graphs), it is natural to generalise the notion of a classical Ramsey number in order to accommodate multipartite graphs by fixing the number of partite sets in the larger multipartite graph whose edges are to be bi-coloured, and then seeking the minimum cardinality of each partite set that would ensure a monochromatic occurrence of some smaller, specified, multipartite subgraph within an arbitrary bi-colouring of the edges of the larger graph. An alternative is of course to fix the cardinality of each partite set in the larger graph and to seek the minimum number of partite sets of this cardinality that would ensure a monochromatic occurrence of the same specified multipartite subgraph.

We denote a complete, balanced, multipartite graph consisting of n partite sets and l vertices per partite set by $K_{n \times l}$, and we consider only diagonal multipartite Ramsey numbers in this paper, although off-diagonal generalisations are also possible.

Definition 1 *Let n, l, k and j be natural numbers. Then the diagonal set multipartite Ramsey number $M_j(n, l)$ is the smallest natural number s such that a bi-colouring of the edges of $K_{s \times j}$ necessarily forces a monochromatic $K_{n \times l}$ as subgraph. The diagonal size multipartite Ramsey number $m_k(n, l)$ is the smallest natural number t such that a bi-colouring of the edges of $K_{k \times t}$ necessarily forces a monochromatic $K_{n \times l}$ as subgraph.*

Note that the subgraphs in Definition 1 need not be vertex-induced subgraphs, i.e. additional edges (of any colour!) may be present between vertices of the same partite set within the forced monochromatic multipartite subgraphs. This definition is a generalisation of diagonal classical Ramsey numbers in the sense that if $r(\sigma, \sigma) = \tau$, then $M_1(\sigma, 1) = \tau$ and $m_\tau(\sigma, 1) = 1$. The off-diagonal classical Ramsey numbers are *not* special cases of our multipartite Ramsey numbers according to the above definition.

After establishing the existence of diagonal multipartite Ramsey numbers according to the above definition in §2, the notion of a circulant graph is introduced in §3, because this class of graphs proves ideal for establishing multipartite Ramsey number lower bounds, as is illustrated in §4. In §5 we present an algorithm which is suitable for determining whether a given graph contains $K_{n \times l}$ as subgraph, and then we apply this algorithm in a computer search for random colourings and circulant colourings which provide lower bounds. Finally, we turn to the question of upper bounds in §6 and conclude the paper with a summary in §7 of certain known multipartite Ramsey numbers as well as best newly obtained bounds for other small numbers.

2 Existence of multipartite Ramsey numbers

The question of the existence of set multipartite Ramsey numbers is settled first, and the method of proof is based upon the known existence of the classical Ramsey numbers. However, in order to prove this existence theorem, we need the notion of an expansive colouring. A colouring of the edges of $K_{k \times j}$ is called an **expansive colouring** if, for every pair of partite sets of $K_{k \times j}$, the edges between all vertices in these partite sets have the same colour. Therefore every expansive colouring of $K_{k \times j}$ corresponds to exactly one edge colouring of K_k (this may be seen by replacing each partite set of $K_{k \times j}$ by a single vertex), and we say that the expansive colouring of $K_{k \times j}$ is induced by the corresponding colouring of K_k (this definition is due to Day, *et al.* [13]).

Theorem 1 *The set multipartite Ramsey number $M_j(n, l)$ exists and, in fact,*

$$\max\{r(n, n), \lceil l/j \rceil n\} \leq M_j(n, l) \leq \binom{2nl - 2}{nl - 1}$$

for all $j, n, l \geq 1$.

Proof: The lower bound is considered first. If $n = 1$, then $r(n, n) = 1$ and hence $\max\{r(n, n), \lceil l/j \rceil n\} = \lceil l/j \rceil$, which is surely a lower bound for $M_j(1, l)$, since a multipartite graph with fewer than $\lceil l/j \rceil$ partite sets and with j vertices per partite set cannot possibly contain K_l as subgraph.

Now suppose that $n \geq 2$ and let $s = r(n, n) \geq 2$. By the definition of s , there exists an edge bi-colouring of K_{s-1} that does not contain a monochromatic K_n as subgraph. The expansive colouring of $K_{(s-1) \times j}$ induced by this bi-colouring therefore also does not contain a monochromatic K_n as subgraph. But $K_n \equiv K_{n \times 1} \subseteq K_{n \times l}$ for any $l \geq 1$, and so the expansive colouring of $K_{(s-1) \times j}$ does not contain a monochromatic $K_{n \times l}$ as subgraph. We conclude that $M_j(n, l) > s - 1$.

In order to establish the second lower bound for $n \geq 2$, we prove that $K_{n \times l} \not\subseteq K_{k \times j}$ if $k < \lceil l/j \rceil n$. Consider first the case where $l \leq j$ and suppose that $k < \lceil l/j \rceil n$. Then $k < n$, but $K_n \subseteq K_{n \times l}$ for any $l \geq 1$, while $K_n \not\subseteq K_{k \times j}$. Therefore $K_{n \times l} \not\subseteq K_{k \times j}$. Consider now the case where $l = mj + r$ for some $m \geq 1$ and $0 \leq r < j$. We show, by attempting to construct n partite sets of a potential subgraph $K_{n \times l}$, that $K_{n \times l} \not\subseteq K_{k \times j}$, unless $k \geq \lceil l/j \rceil n$. Note that two vertices from any partite set of $K_{k \times j}$ cannot occur in different partite sets of an attempted construction of $K_{n \times l}$ within $K_{k \times j}$, since there are no edges between such vertices. Hence we need at least $\lceil l/j \rceil$ partite sets from $K_{k \times j}$ for the construction of a single partite set of $K_{n \times l}$, and there will be $j - r$ superfluous vertices for each such construction. But there must be n partite sets to form $K_{n \times l}$, implying that $k \geq \lceil l/j \rceil n$.

Finally, consider the upper bound. From the existence theorem of Erdős & Szekeres [14] it follows that $r(nl, nl) \leq \binom{2nl-2}{nl-1} = t$, say. Hence, when arbitrarily bi-colouring the edges of K_t , a monochromatic K_{nl} is forced as subgraph. But since $K_{n \times l} \subseteq K_{nl} \subseteq K_t \equiv K_{t \times 1} \subseteq K_{t \times j}$ for any $j \geq 1$, $K_{t \times j}$ necessarily contains a monochromatic $K_{n \times l}$ as subgraph. ■

Although the above existence theorem is constructive in the sense that it provides explicit bounds for $M_j(n, l)$, these bounds are weak, and hence of no practical use, as will be illustrated in §4 & §5. On the other hand, not all size multipartite Ramsey numbers exist (for example, it is clear that $m_1(n, l) = \infty$ if $n > 1$). The following theorem establishes a partial boundedness result for size multipartite Ramsey numbers in terms of the known boundedness of set multipartite Ramsey numbers, due to Theorem 1.

Theorem 2 (1) $m_k(n, l) \geq \lceil nl/k \rceil$ for all $k, n, l \geq 1$.

(2) $m_S(n, l) > j$ if and only if $M_j(n, l) > S$, for all $n, l \geq 1$.

(3) $m_T(n, l) \leq j$ if and only if $M_j(n, l) \leq T$, for all $n, l \geq 1$.

Proof: (1) The graph $K_{n \times l}$ has nl vertices. Hence there must be at least $\lceil nl/k \rceil$ vertices per partite set in a complete, balanced, multipartite graph comprising k partite sets in order to possibly contain $K_{n \times l}$ as subgraph.

(2) The inequality $M_j(n, l) > S$ holds if and only if an arbitrary bi-colouring of the edges of $K_{S \times j}$ does not necessarily contain a monochromatic $K_{n \times l}$ as subgraph, which may be restated as $m_S(n, l) > j$.

(3) The inequality $M_j(n, l) \leq T$ holds if and only if any bi-colouring of the edges of $K_{T \times j}$ necessarily contains a monochromatic $K_{n \times l}$ as subgraph, which is to say if and only if $m_T(n, l) \leq j$. ■

It is also possible to establish the following relationships between set and size multipartite Ramsey numbers.

Proposition 1 *Let k, j, n, l, s and t be natural numbers. Then*

(1) $M_j(s, t) \leq M_j(n, l)$ if $s \leq n$ and $t \leq l$.

(2) $m_k(s, t) \leq m_k(n, l)$ if $s \leq n$ and $t \leq l$.

(3) $M_j(n, l) \leq M_k(n, l)$ if $k \leq j$.

(4) $m_j(n, l) \leq m_k(n, l)$ if $k \leq j$.

Proof: (1) If $M_j(n, l) = r$ then an arbitrary bi-colouring of the edges of $K_{r \times j}$ necessarily contains a monochromatic $K_{n \times l}$ as subgraph. But, since $K_{s \times t} \subseteq K_{n \times l}$ if $s \leq n$ and $t \leq l$, there must also be a monochromatic $K_{s \times t}$ in the (arbitrary) bi-colouring of the edges of $K_{r \times j}$, and hence $M_j(s, t) \leq r$. We conclude that $M_j(s, t) \leq M_j(n, l)$ if $s \leq n$ and $t \leq l$.

(2) The proof of this result is similar to that of part (1).

(3) If $M_k(n, l) = \tau$ then an arbitrary bi-colouring of the edges of $K_{\tau \times k}$ necessarily contains a monochromatic $K_{n \times l}$ as subgraph. But, since $K_{\tau \times k} \subseteq K_{\tau \times j}$ if $k \leq j$, there must also be a monochromatic $K_{n \times l}$ in an arbitrary bi-colouring of the edges of $K_{\tau \times j}$, and hence $M_j(n, l) \leq \tau$. We conclude that $M_j(n, l) \leq M_k(n, l)$ if $k \leq j$.

(4) The proof of this result is similar to that of part (3). ■

Note that there are similar results to those of Proposition 1(1) and 1(2) for the classical Ramsey numbers. The following proposition provides values for the trivial classes of $(1, l)$ and $(n, 1)$ multipartite Ramsey numbers.

- Proposition 2** (1) $M_j(1, l) = \lceil l/j \rceil$ and $m_k(1, l) = \lceil l/k \rceil \forall j, k, l \geq 1$.
- (2) $M_j(n, 1) = r(n, n)$ for all $j, n \geq 1$.
- (3) $m_k(n, 1) = 1$ for all $n \geq 1$ and $k \geq r(n, n)$.
- (4) $m_k(n, l) = \infty$ for all $n \geq 2$ and $l \geq 1, 1 \leq k < r(n, n)$.

Proof: (1) There exists a vertex subset $V^* \subseteq V(K_{\lceil l/j \rceil \times j})$ consisting of l vertices. Now (V^*, \emptyset) constitutes the “monochromatic” graph $K_{1 \times l}$. Therefore $M_j(1, l) \leq \lceil l/j \rceil$, but $M_j(1, l) \geq \lceil l/j \rceil$ by Theorem 1. Consequently $M_j(1, l) = \lceil l/j \rceil$ for all $j, l \geq 1$. It can be shown in a similar way that $m_k(1, l) = \lceil l/k \rceil$ for all $k, l \geq 1$.

(2) $M_j(n, l) \geq r(n, n)$ for all $j, n, l \geq 1$ by Theorem 1, but $M_j(n, 1) \leq M_1(n, 1) = r(n, n)$ by Proposition 1(3). Consequently $M_j(n, 1) = r(n, n)$ for all $j, n \geq 1$.

(3) From classical Ramsey theory we know that $r(n, n) = t$ (say) partite sets is sufficient to force a monochromatic $K_{n \times 1}$ as subgraph of any bi-colouring of the edges of $K_{t \times 1}$. Therefore $m_k(n, 1) \leq 1$ for all $k \geq t$. But then it follows from Definition 1 that $m_k(n, 1) = 1$ for all $k \geq t$.

(4) If $1 \leq k < r(n, n)$ for some $n \geq 2$, then there exists a bi-colouring of the edges of K_k that does not contain a monochromatic K_n as subgraph. But, since $K_n \subseteq K_{n \times l}$ for any $l \geq 1$, the expansive colouring of $K_{k \times j}$ induced by this specific colouring of $E(K_k)$ also does not contain a monochromatic $K_{n \times l}$ as subgraph, no matter how large we choose $j \geq 1$. We conclude that $m_k(n, 1) = \infty$ for all $k < r(n, n)$. ■

Apart from the numbers mentioned in Proposition 2, there are only a few known diagonal multipartite Ramsey numbers. These are $M_1(2, 2) = 6$ due to Chvátal & Harary [9], $M_1(2, 3) = 18$ due to Harborth & Mengersen [22], $m_2(2, 3) = 17$ due to Beineke & Schwenk [1] and the complete class of (2,2) size multipartite Ramsey numbers, as stated in the following theorem.

Theorem 3 (The class of (2,2) size numbers)

- (1) $m_1(2, 2) = \infty$.
- (2) $m_2(2, 2) = 5$.
- (3) $m_3(2, 2) = 3$.
- (4) $m_4(2, 2) = 2$.

$$(5) m_5(2, 2) = 2.$$

$$(6) m_k(2, 2) = 1 \text{ for all } k \geq 6.$$

Part (1) of the above theorem holds by Proposition 2(2), since $r(2, 2) = 2 > 1$. Part (2) of the theorem is due to Beineke & Schwenk [1], while parts (3)–(6) of the theorem are due to Day, *et al.* [13]. The class of (2,2) set Ramsey numbers may now be established.

Proposition 3 (The class of (2,2) set numbers)

$$(1) M_1(2, 2) = 6.$$

$$(2) M_2(2, 2) = 4.$$

$$(3) M_3(2, 2) = 3.$$

$$(4) M_4(2, 2) = 3.$$

$$(5) M_j(2, 2) = 2 \text{ for all } j \geq 5.$$

Proof: (1) This result is due to Chvátal & Harary [9], as stated before.

(2) By Theorem 3(4), $m_4(2, 2) \leq 2$. Therefore it follows by Theorem 2(3) that $M_2(2, 2) \leq 4$. Also, by Theorem 3(3), $m_3(2, 2) > 2$, so that $M_2(2, 2) > 3$ by Theorem 2(2).

(3) By Theorem 3(3), $m_3(2, 2) \leq 3$. Therefore it follows by Theorem 2(3) that $M_3(2, 2) \leq 3$. Also, by Theorem 3(2), $m_2(2, 2) > 3$, so that $M_3(2, 2) > 2$ by Theorem 2(2).

(4) $M_4(2, 2) \leq M_3(2, 2) = 3$ by Proposition 1(3) and Theorem 3(3). Also, by Theorem 3(2), $m_2(2, 2) > 4$, so that $M_4(2, 2) > 2$ by Theorem 2(2).

(5) By Theorem 3(2), $m_2(2, 2) \leq 5$, so that $M_5(2, 2) \leq 2$ by Theorem 2(3). But then $M_j(2, 2) \leq 2$ for all $j \geq 5$ by Proposition 1(3). Furthermore $M_j(2, 2) \neq 1$, because the edge set of $K_{1 \times j}$ is empty for all $j \geq 1$. ■

3 Circulant graphs

In this section we discuss certain basic properties of the class of circulant graphs. Circulants are graphs with a high degree of symmetry and an

elegant structure, which are ideally suited to establish lower bounds for Ramsey numbers. In fact, sharp lower bounds for many of the known classical Ramsey numbers may be attained by using circulants, as illustrated in Figure 3.1.

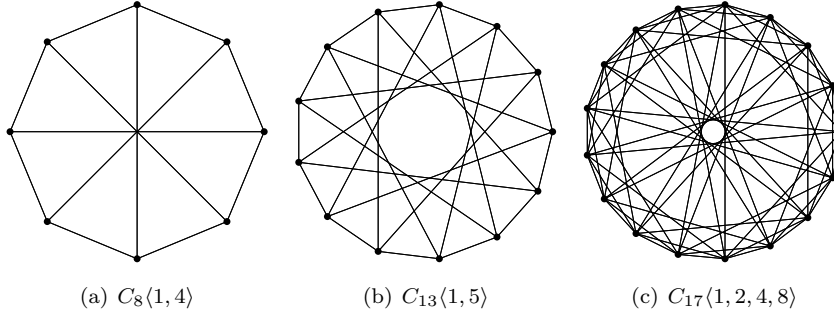


Figure 3.1: (a) $K_3 \not\subseteq C_8\langle 1, 4 \rangle$ and $K_4 \not\subseteq \overline{C_8\langle 1, 4 \rangle} \Rightarrow r(3, 4) > 8$. (b) $K_3 \not\subseteq C_{13}\langle 1, 5 \rangle$ and $K_5 \not\subseteq \overline{C_{13}\langle 1, 5 \rangle} \Rightarrow r(3, 5) > 13$. (c) $K_4 \not\subseteq C_{17}\langle 1, 2, 4, 8 \rangle$ and $K_4 \not\subseteq \overline{C_{17}\langle 1, 2, 4, 8 \rangle} \Rightarrow r(4, 4) > 17$. See, for example [30], p.227.

A circulant is formally defined as follows.

Definition 2 Let T, z be natural numbers with $z < T$ and let $1 \leq i_1, \dots, i_z < T$ be z distinct integers. The **circulant** $C_T\langle i_1, \dots, i_z \rangle$ is the graph with vertex set $V(C_T\langle i_1, \dots, i_z \rangle) = \{v_0, \dots, v_{T-1}\}$ and edge set $E(C_T\langle i_1, \dots, i_z \rangle) = \{v_\alpha v_{\alpha+\beta \pmod{T}} : \alpha = 0, \dots, T-1 \text{ and } \beta = i_1, \dots, i_z\}$. If $z = 1$, then the circulant $C_T\langle i_1 \rangle$ is called an **elementary circulant**; otherwise it is called a **composite circulant**.

Note that

$$C_T\langle i_1, \dots, i_z \rangle = \bigcup_{s=1}^z C_T\langle i_s \rangle, \quad (3.1)$$

i.e. a composite circulant is the union of elementary circulants. Furthermore, since $C_T\langle i_s \rangle = C_T\langle T - i_s \rangle$, we shall assume from now on that $i_s \leq \lfloor T/2 \rfloor$ for all $s = 1, \dots, z$. By this assumption the union in (3.1) is disjoint in the sense that $C_T\langle i_s \rangle$ and $C_T\langle i_r \rangle$ have no edges in common if $s \neq r$. Also note that circulants are regular graphs. Specifically, if $i_s < T/2$ for all $s = 1, \dots, z$, then each elementary circulant $C_T\langle i_s \rangle$ consists of one or more cycles and hence $C_T\langle i_1, \dots, i_z \rangle$ is a $2z$ -regular graph. If $i_s = T/2$, then $C_T\langle i_s \rangle$ is 1-regular, while all other elementary circulants consist of cycles; hence $C_T\langle i_1, \dots, i_s, \dots, i_z \rangle$ is $(2z-1)$ -regular in this case. In the latter

case we call $C_T\langle i_1, \dots, i_s, \dots, i_z \rangle$ a **singular circulant**. A non-singular circulant $C_T\langle i_1, \dots, i_z \rangle$ clearly has zT edges, while a singular circulant of the same form has $(2z - 1)T/2$ edges. It is possible to prove the following useful result.

Proposition 4 $C_{kj}\langle i \rangle \subseteq K_{k \times j}$ if $i \geq j$.

If we assume, for the moment, that the partite sets of $K_{k \times j}$ are given by $V_1 = \{v_0, \dots, v_{j-1}\}$, $V_2 = \{v_j, \dots, v_{2j-1}\}$, \dots , $V_k = \{v_{(k-1)j}, \dots, v_{kj-1}\}$ and denote the graph $C_{kj}\langle i_s \rangle \cap K_{k \times j}$ by $C_{k \times j}\langle i_s \rangle$, then it is clear that $C_{k \times j}\langle i_s \rangle = C_{kj}\langle i_s \rangle$ if and only if $i_s \geq j$. We call the edges within $E(C_{k \times j}\langle j, j+1, \dots, \lfloor kj/2 \rfloor \rangle)$ **ordinary edges** of $K_{k \times j}$. Note that there are also edges in $E(K_{k \times j})$ which are not ordinary edges if $j > 1$. These edges are called the **closure edges** of $K_{k \times j}$. The total number of edges in $K_{k \times j}$ is given by

$$q(K_{k \times j}) = kj^2(k-1)/2 = \underbrace{kj[(k-2)j+1]/2}_{\text{ordinary edges}} + \underbrace{kj(j-1)/2}_{\text{closure edges}}. \quad (3.2)$$

Finally, there are $\binom{\lfloor T/2 \rfloor}{z}$ different circulants of the form $C_T\langle i_1, \dots, i_z \rangle$, but many of these circulants are isomorphic, as we now show. However, we first need a few concepts from elementary abstract algebra. Let $Z_T = \{0, 1, \dots, T-1\}$ be the set of integers modulo T and let M_T be the set of invertible elements of Z_T , i.e. $M_T = \{m \in Z_T : \gcd(m, T) = 1\}$. It is well known that M_T forms a group under multiplication modulo T . For each $m \in M_T$, let

$$\sigma_m = \begin{pmatrix} 0 & 1 & 2 & \dots & T-1 \\ 0 & m & 2m & \dots & m(T-1) \end{pmatrix}$$

and let S_T be the set of permutations of Z_T . Since m is an invertible element of Z_T , it follows that $\sigma_m \in S_T$. Let $\Sigma_T = \{\sigma_m : m \in M_T\}$. Then the group (M_T, \cdot) is isomorphic to the subgroup (Σ_T, \cdot) of (S_T, \cdot) . An isomorphism is a mapping $f : M_T \mapsto \Sigma_T$ such that $f(m) = \sigma_m$. From now on we shall identify M_T with Σ_T , i.e. we shall consider (M_T, \cdot) to be a subgroup of (S_T, \cdot) .

Consider a graph G with vertex set Z_T , edge set E , and let $\sigma \in S_T$. By σG we denote the graph with vertex set Z_T and edge set $\sigma E = \{\sigma(u)\sigma(v) : uv \in E\}$. Clearly, $G \simeq H$ if and only if $H = \sigma G$ for some $\sigma \in S_T$. If $\sigma \in S_T$, then $\sigma C_T\langle i_1, \dots, i_z \rangle$ is not necessarily a circulant. However, if $m \in M_T$, then $m C_T\langle i_1, \dots, i_z \rangle$ is the circulant $C_T\langle mi_1, \dots, mi_z \rangle$. This enables us to generate $|M_T|$ circulants isomorphic to $C_T\langle i_1, \dots, i_z \rangle$, by taking all multiples $m C_T\langle i_1, \dots, i_z \rangle$, $m \in M_T$. There is one problem,

though. We have assumed the entries in our circulants to satisfy $i_s \leq \lfloor T/2 \rfloor$ for all $s = 1, \dots, z$, but it may happen that $mi_s > \lfloor T/2 \rfloor$. To remedy this, let (H_T, \cdot) be the factor group $(M_T/\{1, n-1\}, \cdot)$, i.e. for each $m \in M_T$, we identify m with $T-m$. By taking all multiples $mC_T\langle i_1, \dots, i_z \rangle$, $m \in H_T$, we generate all the circulants mentioned above, but with less duplication and in accordance with our assumption. We remind the reader that $|M_T| = \phi(T)$, where ϕ is the well known Euler function. The following proposition now follows from the above argument.

Proposition 5 *The multiples $mC_T\langle i_1, \dots, i_z \rangle$, $m \in H_T$ generate $\phi(T)/2$ circulants isomorphic to $C_T\langle i_1, \dots, i_z \rangle$.*

This proposition is useful, because it narrows down substantially the amount of candidate circulants that have to be considered when searching for Ramsey number lower bounds.

4 Analytic lower bounds

Using the notion of a circulant graph, it is possible to establish lower bounds, as is illustrated in the following two propositions for respectively the classes of (2,3) and (3,2) set multipartite Ramsey numbers.

Proposition 6 (The class of (2,3)–Ramsey numbers)

- (1) $M_1(2, 3) > 17$ and $m_{17}(2, 3) > 1$.
- (2) $M_2(2, 3) > 7$ and $m_7(2, 3) > 2$.
- (3) $M_3(2, 3) > 5$ and $m_5(2, 3) > 3$.
- (4) $M_4(2, 3) > 3$ and $m_3(2, 3) > 4$.
- (5) $M_j(2, 3) > 1$ for all $j \geq 5$.

Proof: (1) $K_{2 \times 3} \not\subseteq C_{17 \times 1}\langle 3, 5, 6, 7 \rangle$ and $K_{2 \times 3} \not\subseteq \overline{C_{17 \times 1}\langle 3, 5, 6, 7 \rangle} \Rightarrow M_1(2, 3) > 17$ and $m_{17}(2, 3) > 1$. See Figure 4.1(a)&(b). This is, in fact, a sharp lower bound, since it is known that $M_1(2, 3) = 18$, as stated before.

(2) $K_{2 \times 3} \not\subseteq C_{7 \times 2}\langle 2, 3, 5 \rangle$ and $K_{2 \times 3} \not\subseteq C_{7 \times 2}\langle 1, 4, 6, 7 \rangle \Rightarrow M_2(2, 3) > 7$ and $m_7(2, 3) > 2$. See Figure 4.1(c)&(d).

(3) $K_{2 \times 3} \not\subseteq C_{5 \times 3} \langle 3, 4, 6 \rangle$ and $K_{2 \times 3} \not\subseteq C_{5 \times 3} \langle 1, 2, 5, 7 \rangle \Rightarrow M_3(2, 3) > 5$ and $m_5(2, 3) > 3$. See Figure 4.1(e)&(f).

(4) $K_{2 \times 3} \not\subseteq C_{3 \times 4} \langle 1, 2, 4 \rangle$ and $K_{2 \times 3} \not\subseteq C_{3 \times 4} \langle 3, 5, 6 \rangle \Rightarrow M_4(2, 3) > 3$ and $m_3(2, 3) > 4$. See Figure 4.1(g)&(h).

(5) $M_j(2, 3) \geq M_j(2, 1) = r(2, 2) = 2 > 1$ for all $j \geq 5$ by Propositions 1(1) and 2(2). ■

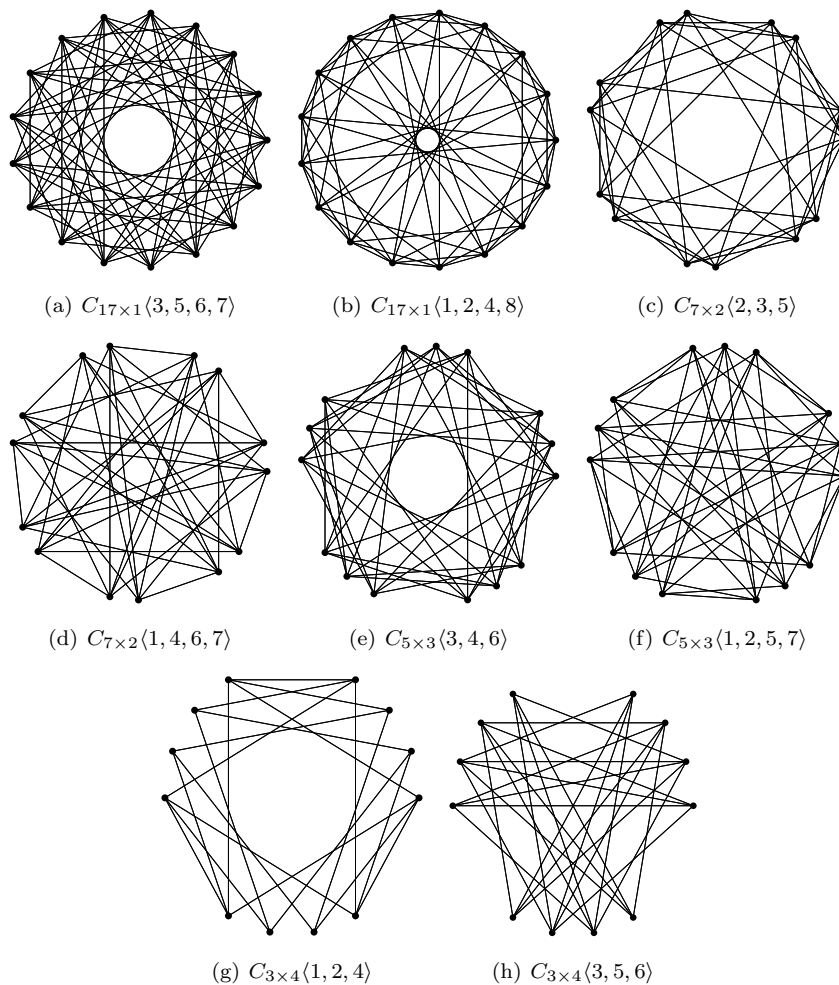


Figure 4.1: Circulants that do not contain $K_{2 \times 3}$ as subgraph.

It is not easy to see that the graphs presented in Figure 4.1 do not contain $K_{2 \times 3}$ as subgraph, but this may be established via tedious arguments involving degree and cycle counts in the graphs presented, and by showing that the properties of these graphs are irreconcilable with those of $K_{2 \times 3}$.

Proposition 7 (The class of $(3, 2)$ -Ramsey numbers)

- (1) $M_1(3, 2) > 29$ and $m_{29}(3, 2) > 1$.
- (2) $M_2(3, 2) > 8$ and $m_8(3, 2) > 2$.
- (3) $M_3(3, 2) > 6$ and $m_6(3, 2) > 3$.
- (4) $M_j(3, 2) > 5$ for all $j \geq 4$.

Proof: (1) $K_{3 \times 2} \not\subseteq C_{29 \times 1} \langle 1, 4, 5, 6, 7, 9, 13 \rangle$ and $K_{3 \times 2} \not\subseteq C_{29 \times 1} \langle 2, 3, 8, 10, 11, 12, 14 \rangle \Rightarrow M_1(3, 2) > 29$ and $m_{29}(3, 2) > 1$. See Figure 4.2(a)&(b).

(2) $K_{3 \times 2} \not\subseteq C_{8 \times 2} \langle 1, 2, 3, 5 \rangle$ and $K_{3 \times 2} \not\subseteq C_{8 \times 2} \langle 4, 6, 7, 8 \rangle \Rightarrow M_2(3, 2) > 8$ and $m_8(3, 2) > 2$. See Figure 4.2(c)&(d).

(3) $K_{3 \times 2} \not\subseteq C_{8 \times 3} \langle 1, 2, 8, 9, 10, 11 \rangle$ and $K_{3 \times 2} \not\subseteq C_{8 \times 3} \langle 3, 4, 5, 6, 7, 12 \rangle \Rightarrow M_2(3, 2) > 8$ and $m_8(3, 2) > 2$. See Figure 4.2(e)&(f).

(4) $M_j(3, 2) \geq M_j(3, 1) = r(3, 3) = 6 > 5$ for all $j \geq 4$ by Propositions 1(1) and 2(2). ■

Again it is not easy to see that the graphs in Figure 4.2 do not contain $K_{3 \times 2}$ as subgraph, but again this may be established by showing that the properties of the graphs are irreconcilable with those of $K_{3 \times 2}$. It is clear, from the tedium of verifying the correctness of the proofs of Propositions 6 & 7, that the computer generation of multipartite Ramsey number lower bounds would be a desirable option to explore.

5 Computer generated lower bounds

When seeking to establish lower bounds for multipartite Ramsey numbers, specific bi-colourings of the edges of certain multipartite graphs which do not contain certain other monochromatic multipartite graphs as subgraphs have to be produced, as was illustrated in the previous section. In order to compute these specific bi-colourings, one of two approaches may be used:

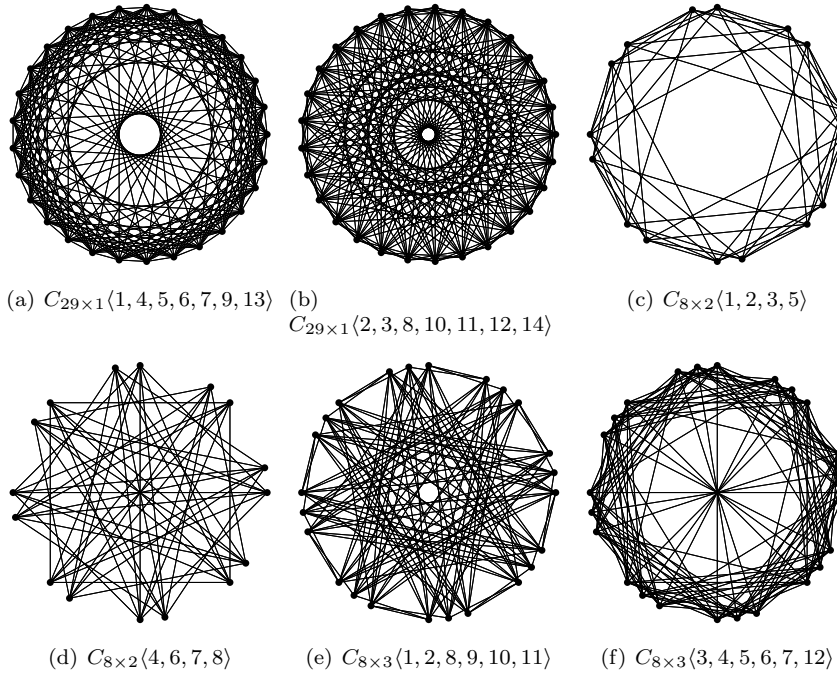


Figure 4.2: Circulants that do not contain $K_{3 \times 2}$ as subgraph.

one may attempt to produce two multipartite graph complement pseudo-random bi-colourings of the edges of a multipartite graph and hope to find a specific colouring which does not contain a desired monochromatic multipartite graph as subgraph. An alternative approach would be the attempted use of known properties of a certain class of graphs, and a colouring of the edges of a multipartite graph according to the structure of this class, in the hope of finding a specific colouring according to a member of the class which does not contain a desired monochromatic multipartite graph as subgraph. In this section we employ both methods: pseudo-random colourings as well as colourings according to circulant graph structures. However, in both cases one needs a rule by which to decide whether a multipartite graph occurs as subgraph of a given graph. Unfortunately the task of determining whether $K_{n \times l}$ occurs as subgraph of another given graph G is an NP-complete problem¹. Nevertheless, when seeking to establish lower

¹Garey & Johnson [19] proved that, given a graph G and a positive integer $c \leq |V(G)|$, the problem of deciding whether there are two disjoint subsets $V_1, V_2 \subseteq V(G)$ such that $|V_1| = |V_2| = c$ and such that $u \in V_1, v \in V_2$ implies that $uv \in E(G)$, is NP-complete.

bounds for *small* Ramsey numbers, the following algorithm is practical for establishing whether $K_{n \times l}$ occurs as subgraph of a given graph G .

DESCRIPTION OF THE ALGORITHM

There are a number of possible reasons why a given graph G might not contain the multipartite graph $K_{n \times l}$ as subgraph:

- (1) G may have too few vertices or too few edges to contain $K_{n \times l}$ as subgraph.
- (2) Or, if G has enough vertices and edges, G may have too few vertices with large enough neighbourhood sets to contain $K_{n \times l}$ as subgraph.
- (3) Or, if G has enough vertices with large enough neighbourhood sets, G may not contain at least n sets of vertices such that each vertex in a set is adjacent to $(n - 1)l$ mutual neighbours.
- (4) Or, if G contains at least n such sets of vertices, no combination of n of these sets may be mutually disjoint, or there might be a combination of n of these sets that are in fact mutually disjoint, but the mutual neighbours of some set in the combination might be outside the specific combination of n sets.

These four criteria are not mutually exclusive, and have been ordered in such a way as to minimise, on average, the computational complexity of testing whether $K_{n \times l} \subseteq G$. To render our algorithm as efficient as possible, our approach is therefore to suspect from the outset that $K_{n \times l} \not\subseteq G$, and to try and prove this suspicion in the cheapest possible fashion (by testing the above four criteria in the above mentioned order until success is achieved), or to conclude that our suspicion was wrong.

Consequently, the strategy behind the algorithm is first to check whether the given graph G is short of the required nl vertices and $nl^2(n - 1)/2$ edges; if this test succeeds, then $K_{n \times l} \not\subseteq G$. Secondly, if this test fails, then the row indices of the adjacency matrix \mathbf{A} of G corresponding to those vertices with neighbourhood sets of size at least the required $(n - 1)l$ are stored in the vector \mathcal{N} . If \mathcal{N} contains fewer than nl entries, then $K_{n \times l} \not\subseteq G$. Thirdly, if this test fails, all possible l -tuples are identified from amongst the entries of \mathcal{N} . For each such l -tuple it is determined whether the corresponding set of l vertices of G has at least $(n - 1)l$ mutual neighbours. This is done by testing whether there are at least $(n - 1)l$ column sums of those l rows of the adjacency matrix corresponding to the vertices represented by the particular l -tuple that equals l . Two matrices are gradually constructed during an application of the algorithm: a matrix

\mathbf{B} is built up, consisting of rows representing those l -tuples of vertices that have the required $(n-1)l$ mutual neighbours (each such row of \mathbf{B} contains ones in positions of the vertices within the particular l -tuple that has the required number of mutual neighbours, and zeros in all other positions). A matrix \mathbf{C} is built up simultaneously, consisting of rows representing the required $(n-1)l$ mutual neighbours of the l -tuples stored in \mathbf{B} : each such row of \mathbf{C} contains ones in positions of mutual neighbours of the particular l -tuple of vertices in question, and zeros in all other positions. If, after considering all possible l -tuples from amongst the entries of \mathcal{N} , the matrix \mathbf{B} contains fewer than n rows, then $K_{n \times l} \not\subseteq G$. Fourthly, if this test fails, all possible n -tuples from the row indices of \mathbf{B} and their corresponding neighbours in \mathbf{C} are identified. Every such n -tuple represents a set of n vertex subsets of G , each with large enough mutual neighbourhood sets to possibly induce a graph containing $K_{n \times l}$ as subgraph. If, for each such n -tuple, there are fewer than nl sums of those rows of \mathbf{B} corresponding to the specific n -tuple that add up to exactly 1, or if there are fewer than $(n-1)l$ sums of those rows of \mathbf{C} corresponding to the specific n -tuple that add up to exactly 1, then $K_{n \times l} \not\subseteq G$, since for every choice of n (potentially partite) vertex subsets of G with size l and mutual neighbourhood set, the sets are either not mutually disjoint, or else not all the (mutual) neighbours of vertices in every set coincide with vertices from the other (potentially partite) vertex subsets. If this test also fails, then we must conclude that indeed $K_{n \times l} \subseteq G$.

In the algorithm the following notation will be used: $\underline{e}(m)$ denotes a row vector of length m containing only ones as entries. If \underline{i}^l is an l -tuple, then $\underline{e}(\underline{i}^l)$ represents a row vector with ones in positions corresponding to the entries of \underline{i}^l and zeros in all other positions, while $\mathbf{A}(\underline{i}^l)$ denotes the submatrix of l rows corresponding to the entries of the l -tuple \underline{i}^l from a given matrix \mathbf{A} .

ALGORITHM 5.1 [To determine whether $K_{n \times l} \subseteq G$.]

Input: The order p , size q and adjacency matrix \mathbf{A} of G ; positive integers n and l .

Output: The boolean variable SUBGRAPH which takes the value **true** if $K_{n \times l} \subseteq G$ and **false** otherwise.

- (1) If $p < nl$ or $q < nl^2(n-1)/2$, then SUBGRAPH := **false**. Stop.
- (2) Let $\mathcal{N} := \{N_1, \dots, N_t\}$ be the set of row indices in \mathbf{A} with a row sum of at least $(n-1)l$. If $t < nl$, then SUBGRAPH := **false**. Stop.
- (3) (a) Let $\alpha := 0$, $\beta := \binom{t}{l}$ and let $\underline{i}_1^l, \underline{i}_2^l, \dots, \underline{i}_\beta^l$ be all possible l -tuples from \mathcal{N} .

- (b) Repeat for all $s := 1, \dots, \beta$: If the vector $[\underline{e}(l)\mathbf{A}(\underline{i}_s^l)]$ (div l) contains at least $(n-1)l$ ones, then let $\mathbf{B}(s) := \underline{e}(\underline{i}_s^l)$ and $\mathbf{C}(s) := [\underline{e}(l)\mathbf{A}(\underline{i}_s^l)]$ (div l), and increment α by one.
- (c) If $\alpha < n$ then SUBGRAPH := **false**. Stop.
- (4) (a) Let $\gamma := \binom{\alpha}{n}$ and let $\underline{\kappa}_1^n, \underline{\kappa}_2^n, \dots, \underline{\kappa}_\gamma^n$ be all possible n -tuples from $\{1, \dots, \alpha\}$.
- (b) If there exists an $s \in \{1, \dots, \gamma\}$ such that $\underline{e}(n)\mathbf{B}(\underline{\kappa}_s^n)$ contains exactly nl ones and $[\underline{e}(n)\mathbf{C}(\underline{\kappa}_s^n)]$ (div $(n-1)$) contains exactly $(n-1)l$ ones, then SUBGRAPH := **true**; else SUBGRAPH := **false**. Stop.

This algorithm is only practical for small values of n , l and p , since a worst case complexity scenario would involve the successive computation in steps 3(b) and 4(b) respectively of $\binom{p}{l} = \delta$ (say) row & column combinations, followed by $\binom{\delta}{n}$ column sum computations. Even though the algorithm becomes dramatically inefficient as n and/or l increases, the use of the algorithm may still be more desirable than exhaustive hand searches for multipartite graph colourings, such as those presented in §4. The following theorem validates Algorithm 5.1.

Theorem 4 *The boolean variable in Algorithm 5.1 satisfies SUBGRAPH = true if and only if $K_{n \times l} \subseteq G$.*

Proof: Suppose SUBGRAPH = **true** after an application of Algorithm 5.1. Then there exists an $s \in \{1, \dots, \gamma\}$ such that the row vector $\underline{e}(n)\mathbf{B}(\underline{\kappa}_s^n)$ contains exactly nl ones and the row vector $\underline{e}(n)\mathbf{C}(\underline{\kappa}_s^n)$ contains exactly $(n-1)l$ entries of magnitude $n-1$ by step 4(b) of the algorithm. But $\underline{e}(n)\mathbf{B}(\underline{\kappa}_s^n)$ and $\underline{e}(n)\mathbf{C}(\underline{\kappa}_s^n)$ represent the column sums of those rows of the matrices \mathbf{B} and \mathbf{C} respectively whose indices correspond to the n -tuple $\underline{\kappa}_s^n$. Each row of the submatrix $\mathbf{B}(\underline{\kappa}_s^n)$ in turn represents an l -set of vertices with mutual neighbours, which are in turn represented by the corresponding rows of the submatrix $\mathbf{C}(\underline{\kappa}_s^n)$. Because $\underline{e}(n)\mathbf{B}(\underline{\kappa}_s^n)$ contains exactly nl ones according to step 4(b), there exist n l -tuples $\underline{i}_{s_1}^l, \underline{i}_{s_2}^l, \dots, \underline{i}_{s_n}^l$ (say), such that the row vectors $\underline{e}(l)\mathbf{A}(\underline{i}_{s_j}^l)$ contain at least $(n-1)l$ entries of magnitude l for all $j = 1, \dots, n$, so that $\alpha \geq n$ in step 3(c). Now each of these l -tuples represents a set of l vertices in G that have at least $(n-1)l$ mutual neighbours according to the adjacency matrix \mathbf{A} . But since $\underline{e}(n)\mathbf{C}(\underline{\kappa}_s^n)$ contains exactly $(n-1)l$ entries of magnitude $n-1$ according to step 4(b), the mutual neighbours of any such l -set of vertices are amongst the vertices of the other $n-1$ l -sets. Hence a copy of $K_{n \times l}$ exists as subgraph of G .

Conversely, suppose $K_{n \times l}$ occurs as subgraph of an order p graph G with size q and adjacency matrix \mathbf{A} . Label the vertices of G such that $P_i = \{v_{(i-1)l+1}, v_{(i-1)l+2}, \dots, v_{il}\}$ ($i = 1, \dots, n$) are the partite sets of the occurrence of $K_{n \times l}$. Then surely $p \geq nl$ and $q \geq nl^2(n-1)/2$ in step 1 of the algorithm, and since each vertex $v_{(i-1)l+j} \in P_i$ is adjacent all $(n-1)l$ vertices in the partite sets P_k ($k \neq i$), the set \mathcal{N} in step 2 contains at least the $(n-1)l$ entries $(i-1)l+j$ ($i = 1, \dots, n$ and $j = 1, \dots, l$), so that $t \geq nl$. Furthermore the l -tuples $((i-1)l+1, \dots, il)$, occur amongst the set $\{i_1^l, \dots, i_\beta^l\}$ in step 3(a) of the algorithm, and since the row vector $\underline{e}(l)\mathbf{A}((i-1)l+1, \dots, il)$ (div l) contains unit entries in those positions corresponding to the vertices that are mutual neighbours of $v_{(i-1)l+1}, \dots, v_{il}$, it follows by the definition of a partite set that the vector $[\underline{e}(l)\mathbf{A}((i-1)l+1, \dots, il)]$ (div l) contains at least $(n-1)l$ unit entries and hence that the matrix \mathbf{B} contains the corresponding rows $\underline{e}((i-1)l+1, \dots, il)$ for all $i = 1, \dots, n$. Consequently $\alpha \geq n$ in step 3(c) and $\underline{\kappa}_s^n = (1, 2, \dots, n)$ is amongst the n -tuples $\{\underline{\kappa}_1^n, \dots, \underline{\kappa}_\gamma^n\}$ in step 4(a) of the algorithm. This means that the first nl entries of the row vector $\underline{e}(n)\mathbf{B}(\underline{\kappa}_s^n)$ are ones and the first $(n-1)l$ entries of the row vector $\underline{e}(n)\mathbf{C}(\underline{\kappa}_s^n)$ have magnitude $n-1$, resulting in the boolean value `SUBGRAPH = true`. ■

Lower bounds via random edge colourings

Pseudo-random colourings of the edges of the multipartite graph $K_{k \times j}$ were produced in the following manner. An arbitrary vertex v of $K_{k \times j}$ was chosen and the neighbourhood set $N(v)$, consisting of all the neighbours of v that are connected to v by yet uncoloured edges, was isolated. A vertex w was then chosen randomly from $N(v)$ and the edge vw was coloured red. The vertex w was removed from the set $N(v)$ and another vertex w' was chosen from the remaining set $N(v)$. This time the edge vw' was coloured blue, and the vertex w' was removed from $N(v)$. This procedure was repeated, colouring the resulting edges alternatively red and blue until the set $N(v)$ was empty. The whole process was then repeated for all vertices $v \in K_{k \times j}$ to obtain a pseudo-random colouring of $K_{k \times j}$, where the numbers of blue and red edges differ by at most 1, because it was anticipated that bi-colourings of the edges of $K_{k \times j}$ with roughly the same number of red edges as blue edges would be most suitable for finding lower bounds. Algorithm 5.1 was then applied to the subgraphs induced by the resulting colouring in order to determine whether $K_{n \times l}$ occurred as subgraph of these graphs. If $K_{n \times l}$ occurred in neither of the subgraphs induced by a specific colouring, the lower bounds $M_j(n, l) \geq k$ and $m_k(n, l) \geq j$ were established, after which the values for k and j were increased respectively. Some of the

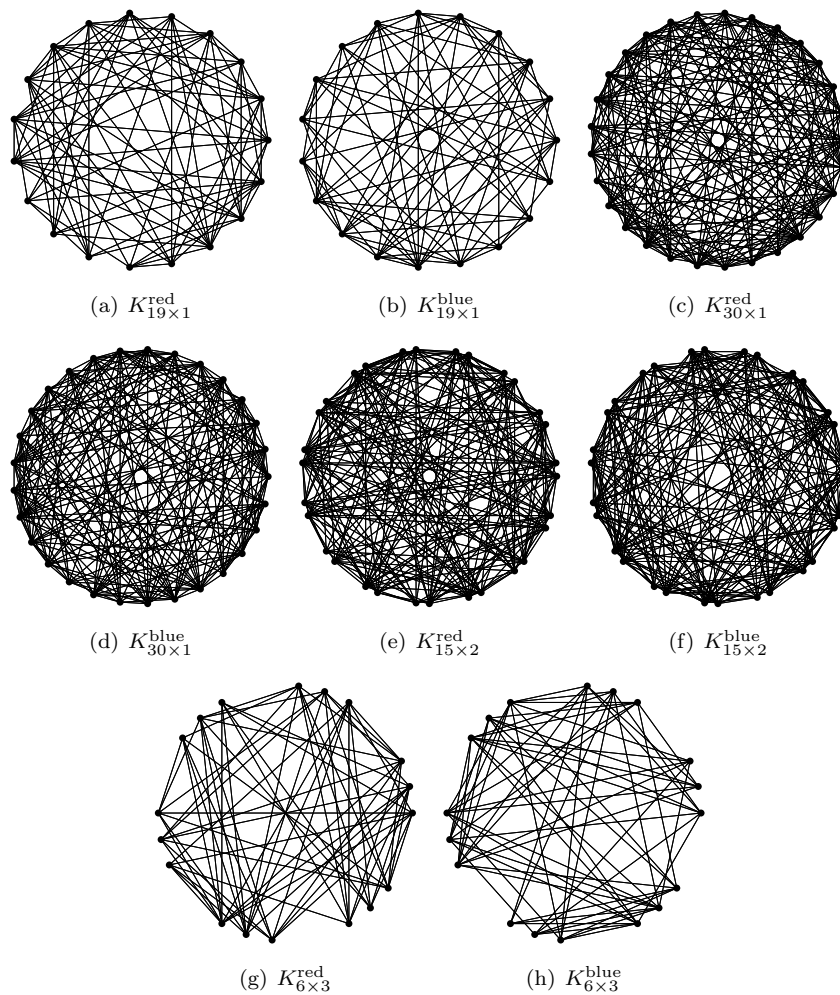


Figure 5.1: Lower bounds for set multipartite numbers using pseudo-random colourings. 1. $K_{2 \times 4} \not\subseteq K_{19 \times 1}^{\text{red}}$ and $K_{2 \times 4} \not\subseteq K_{19 \times 1}^{\text{blue}} \Rightarrow M_1(2, 4) > 19$. 2. $K_{3 \times 3} \not\subseteq K_{30 \times 1}^{\text{red}}$ and $K_{3 \times 3} \not\subseteq K_{30 \times 1}^{\text{blue}} \Rightarrow M_1(3, 3) > 30$. 3. $K_{3 \times 3} \not\subseteq K_{15 \times 2}^{\text{red}}$ and $K_{3 \times 3} \not\subseteq K_{15 \times 2}^{\text{blue}} \Rightarrow M_2(3, 3) > 15$. 4. $K_{3 \times 2} \not\subseteq K_{6 \times 3}^{\text{red}}$ and $K_{3 \times 2} \not\subseteq K_{6 \times 3}^{\text{blue}} \Rightarrow M_3(3, 2) > 6$.

results, after up to a maximum of 10 000 random trial colourings per fixed k and j , are shown in Figure 5.1, while a more complete listing of lower bounds found in this manner for small multipartite Ramsey numbers may be found in Table 5.1.

Lower bounds for $M_j(n, l)$					Lower bounds for $m_k(n, l)$				
		$l = 2$	$l = 3$	$l = 4$			$l = 2$	$l = 3$	$l = 4$
$j = 1$	$n = 2$	6	13	20	$k = 1$	$n = 2$	*	*	*
	$n = 3$	16	31	48		$n = 3$	*	*	*
	$n = 4$	20	42	73		$n = 4$	*	*	*
$j = 2$	$n = 2$	4	7	11	$k = 2$	$n = 2$	*	6	16
	$n = 3$	9	16	26		$n = 3$	*	*	*
	$n = 4$	*	21	39		$n = 4$	*	*	*
$j = 3$	$n = 2$	3	5	8	$k = 3$	$n = 2$	*	6	10
	$n = 3$	7	12	18		$n = 3$	*	*	*
	$n = 4$	*	*	26		$n = 4$	*	*	*

Table 5.1: Lower bounds for small set multipartite Ramsey numbers, as found after 10 000 pseudo-random colourings. An asterisk means that better results were obtained from the theoretical bounds in §2 than was possible via pseudo-random colourings.

Lower bounds via circulant edge colourings

Because it was expected that bi-colourings of $K_{k \times j}$ with roughly the same number of red edges as blue edges would be most suitable for finding lower bounds, circulants of the form $C_{k \times j}^{\text{red}} \langle i_1, \dots, i_{z^*} \rangle$ and $C_{k \times j}^{\text{blue}} \langle j_1, \dots, j_{z^*} \rangle$ were used as edge colourings of $K_{k \times j}$, where z^* was taken as close as possible to $\lfloor kj/4 \rfloor$, while still ensuring that the resulting circulants remain multipartite graph complements. Algorithm 5.1 was then applied to each of these edge-colourings in search of lower bounds. The resulting lower bounds were generally found to be better than those obtained via pseudo-random colourings (after up to 10 000 trial colourings per fixed k and j) for small values of n and l . For larger values of n and l , lower bounds were generally more readily obtained via pseudo-random colourings than via circulant colourings. The reason for this is that suitable values for k and j in the circulants $C_{k \times j}^{\text{red}} \langle i_1, \dots, i_{z^*} \rangle$ and $C_{k \times j}^{\text{blue}} \langle j_1, \dots, j_{z^*} \rangle$ are dictated by the values of n and l when searching for lower bounds for $M_j(n, l)$ and $m_k(n, l)$. For small values of n and l all circulants of the forms $C_{k \times j}^{\text{red}} \langle i_1, \dots, i_{z^*} \rangle$ and $C_{k \times j}^{\text{blue}} \langle j_1, \dots, j_{z^*} \rangle$ could be tested to see whether they contained $K_{n \times l}$ as subgraph, but as the values of n and l increase, so do those of k and j , and it became very expensive to search through all the relevant non-isomorphic circulant classes, even with the benefit of Proposition 5. Some of the lower

bounds for small Ramsey numbers, found via circulant edge colourings, are shown in Table 5.2.

Lower bounds for $M_j(n, l)$					Lower bounds for $m_k(n, l)$				
		$l = 2$	$l = 3$	$l = 4$			$l = 2$	$l = 3$	$l = 4$
$j = 1$	$n = 2$	6	18	30	$k = 1$	$n = 2$	*	*	*
	$n = 3$	30	38	54		$n = 3$	*	*	*
	$n = 4$	30	38	48		$n = 4$	*	*	*
$j = 2$	$n = 2$	4	8	14	$k = 2$	$n = 2$	5	17	18
	$n = 3$	9	17	27		$n = 3$	*	*	*
	$n = 4$	*	21	23		$n = 4$	*	*	*
$j = 3$	$n = 2$	3	6	10	$k = 3$	$n = 2$	3	9	13
	$n = 3$	7	13	18		$n = 3$	*	*	*
	$n = 4$	*	*	18		$n = 4$	*	*	*

Table 5.2: Lower bounds for small set multipartite Ramsey numbers, as found by circulant colourings. An asterisk means that better results were obtained from the theoretical bounds in §2 than was possible via circulant colourings. The details of which circulants were used to obtain the results listed in this table will be given in the conclusion.

6 Upper bounds

Upper bounds for multipartite Ramsey numbers are far more difficult to determine than is the case for lower bounds. Of course there are the upper bounds presented in Theorem 1 to fall back upon, but these bounds are weak, as was mentioned earlier. In this section we present arguments which establish sharper upper bounds for the special case where the sought after monochromatic subgraph is bipartite.

However, we first need the notion of a monochromatic l -connection. Suppose G is a graph of order at least $l + 1$, then a set of l vertices within $V(G)$ which are all neighbours of some vertex $v_0 \in V(G)$ and which are connected to v_0 by means of monochromatic edges is referred to as a monochromatic l -connection of v_0 within G . Using this notion, it is possible to establish the following bipartite Ramsey number upper bounds.

Theorem 5 $M_j(2, l) \leq \left\lceil \frac{2l-1}{j} \right\rceil + \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j} \right\rceil$ for all $j, l \geq 1$.

Proof: Let

$$c = \left\lceil \frac{2l-1}{j} \right\rceil + \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j} \right\rceil$$

and suppose (S, T) is a bi-partition of the partite sets of $K_{c \times j}$, with $|T| = \lceil (2l-1)/j \rceil$. Now let U be any subset of $2(l-1)\binom{2l-1}{l} + 1$ vertices in S and let W be any subset of $2l-1$ vertices in T . Since any two vertices $u \in U$ and $w \in W$ are from different partite sets of $K_{c \times j}$, it follows that $uw \in E(K_{c \times j})$. If we therefore only consider the edges between vertices in U and vertices in W , the degree of every vertex in U is $2l-1$ and either the red or blue degree of every vertex in W is greater than or equal to l . Hence every one of the $2(l-1)\binom{2l-1}{l} + 1$ vertices in U possesses a monochromatic l -connection in any bi-colouring of the edges of $K_{c \times j}$. By the pigeonhole principle, when these $2(l-1)\binom{2l-1}{l} + 1$ monochromatic l -connections are assigned to $\binom{2l}{l}$ subsets of size l in W , there is at least one subset with $2(l-1) + 1$ (or more) monochromatic l -connections assigned to it. But then at least l of these $2l-1$ monochromatic l -connections are red (say), inducing a red $K_{2 \times l}$ as subgraph of $K_{c \times j}$. ■

Although the upper bound in Theorem 5 is better than that of Erdős & Szekeres, as stated in Theorem 1, the upper bound is still not sharp. This may be seen from the example that while $M_1(2, 2) = 6$, Theorem 5 merely states that $M_1(2, 2) \leq 10$. Theorem 1, on the other hand, states that $M_1(2, 2) \leq 20$. The difference in the upper bounds presented in Theorems 1 and 5 of course becomes more dramatic as j or l increases, as is illustrated in Table 6.1.

	$l = 1$		$l = 2$		$l = 3$		$l = 4$	
$j = 1$	2	<i>2</i>	20	<i>10</i>	252	<i>46</i>	3 432	<i>218</i>
$j = 2$	2	<i>2</i>	20	<i>6</i>	252	<i>24</i>	3 432	<i>110</i>
$j = 3$	2	<i>2</i>	20	<i>4</i>	252	<i>16</i>	3 432	<i>74</i>
$j = 4$	2	<i>2</i>	20	<i>3</i>	252	<i>13</i>	3 432	<i>55</i>

Table 6.1: Upper bounds for the set Ramsey numbers $M_j(2, l)$ for $j, l = 1, 2, 3, 4$. The bold faced bounds are by Theorem 1, while italicised bounds are by Theorem 5.

Using a different notation, Hattingh & Henning [25] proved a slightly sharper upper bound for size bipartite Ramsey numbers than that obtained via the bounds of Erdős & Szekeres, when applied directly to bipartite graphs, as is stated in the following theorem.

Theorem 6 $m_2(2, l) \leq \binom{2l}{l} - 1$ for all $l \geq 1$.

The following generalisation of their result is possible.

Theorem 7 $m_k(2, l) \leq \max \left\{ 2l - 1, \left\lceil \frac{2(l-1)\binom{2l-1}{l} + 1}{k-1} \right\rceil \right\}, k \geq 2, l \geq 1.$

Proof: Let

$$c = \max \left\{ 2l - 1, \left\lceil \frac{2(l-1)\binom{2l-1}{l} + 1}{k-1} \right\rceil \right\}$$

and let (S', T') be a bi-partition of the partite sets of $K_{k \times c}$, where $|S'| = k - 1$. Since $c \geq \left\lceil \frac{2(l-1)\binom{2l-1}{l} + 1}{k-1} \right\rceil$ and $c \geq 2l - 1$, the set S' contains at least $2(l-1)\binom{2l-1}{l} + 1$ vertices, while T' contains at least $2l - 1$ vertices. Let U' be any subset of $2(l-1)\binom{2l-1}{l} + 1$ vertices of S' and let W' be any subset of $2l - 1$ vertices of T' . Since any two vertices $u \in U'$ and $w \in W'$ are from different partite sets of $K_{k \times c}$, it follows that $uw \in E(K_{k \times c})$. The rest of the proof is similar to that of Theorem 5, where the argument is repeated with U' instead of U and W' instead of W . ■

Note that although Theorem 7 is more general than Theorem 6, it is also weaker than Theorem 6 for the special case where $k = 2$. This is because the convenient bipartite structure of the larger graph to be coloured cannot be exploited in Theorem 7.

7 Conclusions

In this paper the notion of a classical Ramsey number was generalised by replacing the requirement of a complete graph in the definition by that of a complete, balanced, multipartite graph. The existence of these generalised Ramsey numbers was proved and lower as well as upper bounds for these numbers were established. These results are summarised in Tables 7.1 and 7.2 for small multipartite Ramsey numbers.

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$M_j(n, l)$		$l = 1$	$l = 2$	$l = 3$	$l = 4$
$j = 1$	$n = 1$	1 ¹	2 ¹	3 ¹	4 ¹
	$n = 2$	2 ²	6 ³	18 ⁴	$(30^5:218^a)$
	$n = 3$	6 ⁶	$(30^7:165^b)$	$(38^8:6588^c)$	$(54^9:\star^d)$
	$n = 4$	18 ¹⁰	$(30^{11}:1870^f)$	$(42^{12}:\star^g)$	$(73^{13}:\star^h)$
$j = 2$	$n = 1$	1 ¹	1 ¹	2 ¹	2 ¹
	$n = 2$	2 ¹⁴	4 ³	$(8^{15}:18^e)$	$(14^{16}:110^a)$
	$n = 3$	6 ¹⁴	$(9^{17}:165^e)$	$(17^{18}:6588^e)$	$(27^{19}:\star^d)$
	$n = 4$	18 ¹⁴	$(18^{20}:1870^e)$	$(21^{21}:\star^g)$	$(39^{22}:\star^h)$
$j = 3$	$n = 1$	1 ¹	1 ¹	1 ¹	2 ¹
	$n = 2$	2 ¹⁴	3 ³	$(6^{23}:16^a)$	$(10^{24}:74^a)$
	$n = 3$	6 ¹⁴	$(7^{25}:165^e)$	$(13^{26}:6588^e)$	$(18^{27}:\star^d)$
	$n = 4$	18 ¹⁴	$(18^{20}:1870^e)$	$(18^{20}:\star^g)$	$(26^{28}:\star^h)$
$j = 4$	$n = 1$	1 ¹	1 ¹	1 ¹	1 ¹
	$n = 2$	2 ¹⁴	3 ³	$(4^{29}:13^a)$	$(7^{30}:55^a)$
	$n = 3$	6 ¹⁴	$(6^{20}:165^e)$	$(9^{31}:6588^e)$	$(14^{32}:\star^d)$
	$n = 4$	18 ¹⁴	$(18^{20}:1870^e)$	$(18^{20}:\star^g)$	$(20^{33}:\star^h)$

Table 7.1: Bounds for diagonal set multipartite Ramsey numbers $M_j(n, l)$ for $n, l, j = 1, 2, 3, 4$. The known Ramsey numbers are typeset in bold-faced characters. The bounds for the unknown numbers are inclusive and are presented in the form (upper bound : lower bound). The known Ramsey numbers and lower bounds are motivated as follows: ¹By Proposition 2(1). ²Classical Ramsey number (see Chartrand & Oellermann, [6]). ³By Proposition 3. ⁴Due to Harborth & Mengersen [22]. ⁵ $C_{29 \times 1} \langle 1, 4, 5, 6, 7, 9, 13 \rangle$. ⁶Classical Ramsey number, due to Greenwood & Gleason, [21]. ⁷ $C_{29 \times 1} \langle 1, 4, 5, 6, 7, 9, 13 \rangle$. ⁸ $C_{37 \times 1} \langle 1, 2, 4, 5, 7, 10, 11, 12, 18 \rangle$. ⁹ $C_{53 \times 1} \langle 2, 4, 5, 6, 8, 13, 15, 16, 21, 22, 24, 25, 26 \rangle$. ¹⁰Classical Ramsey number, due to Greenwood & Gleason, [21]. ¹¹ $C_{29 \times 1} \langle 1, 2, 4, 5, 6, 7, 8 \rangle$. ¹²Random colouring. ¹³Random colouring. ¹⁴By Proposition 2(2). ¹⁵ $C_{7 \times 2} \langle 2, 3, 5 \rangle$. ¹⁶ $C_{13 \times 2} \langle 3, 4, 5, 6, 8, 11 \rangle$. ¹⁷ $C_{8 \times 2} \langle 1, 2, 3, 5 \rangle$. ¹⁸ $C_{16 \times 2} \langle 1, 2, 3, 4, 5, 6, 8, 9 \rangle$. ¹⁹ $C_{26 \times 2} \langle 1, 5, 6, 7, 9, 11, 13, 14, 16, 19, 21, 22, 25 \rangle$. ²⁰By Proposition 1(1). ²¹ $C_{20 \times 2} \langle 1, 3, 4, 5, 6, 8, 9, 11, 12, 19 \rangle$. ²²Random colouring. ²³ $C_{5 \times 3} \langle 3, 4, 6 \rangle$. ²⁴ $C_{9 \times 3} \langle 3, 4, 6, 8, 9, 10 \rangle$. ²⁵ $C_{8 \times 3} \langle 3, 4, 5, 6, 7, 12 \rangle$. ²⁶ $C_{12 \times 3} \langle 1, 2, 3, 4, 5, 7, 8, 9, 10 \rangle$. ²⁷ $C_{17 \times 3} \langle 2, 6, 8, 9, 10, 12, 13, 16, 19, 21, 24, 25 \rangle$. ²⁸Random colouring. ²⁹ $C_{3 \times 4} \langle 1, 2, 4 \rangle$. ³⁰ $C_{6 \times 4} \langle 1, 2, 4, 7, 8, 9 \rangle$. ³¹ $C_{8 \times 4} \langle 1, 2, 3, 4, 5, 6, 7, 8 \rangle$. ³² $C_{13 \times 4} \langle 2, 6, 8, 12, 14, 17, 18, 19, 20, 21, 22, 25, 26 \rangle$. ³³Random colouring. The upper bounds are motivated as follows: ^aBy Theorem 5. ^bUpper bound of $r(6, 6) \leq 1870$. ^cUpper bound of $r(9, 9) \leq 6588$. ^{d, g}Upper bound of 705432 by Theorem 1. ^eBy Proposition 1. ^fUpper bound of $r(8, 8) \leq 1870$. ^hUpper bound of 155117520 by Theorem 1. (An asterisk represents an upper bound which is too large to include in the table)

$m_k(n, l)$		$l = 1$	$l = 2$	$l = 3$	$l = 4$
$k = 1$	$n = 1$	1 ¹	2 ¹	3 ¹	4 ¹
	$n = 2$	∞^2	∞^2	∞^2	∞^2
	$n = 3$	∞^2	∞^2	∞^2	∞^2
	$n = 4$	∞^2	∞^2	∞^2	∞^2
$k = 2$	$n = 1$	1 ¹	1 ¹	2 ¹	2 ¹
	$n = 2$	1 ³	5 ⁴	17 ⁴	(18 ⁵ :70 ^a)
	$n = 3$	∞^2	∞^2	∞^2	∞^2
	$n = 4$	∞^2	∞^2	∞^2	∞^2
$k = 3$	$n = 1$	1 ¹	1 ¹	1 ¹	2 ¹
	$n = 2$	1 ³	3 ⁶	(9 ⁷ :17 ^b)	(13 ⁸ :70 ^b)
	$n = 3$	∞^2	∞^2	∞^2	∞^2
	$n = 4$	∞^2	∞^2	∞^2	∞^2
$k = 4$	$n = 1$	1 ¹	1 ¹	1 ¹	1 ¹
	$n = 2$	1 ³	2 ⁶	(6 ⁹ :14 ^c)	(9 ¹⁰ :70 ^b)
	$n = 3$	∞^2	∞^2	∞^2	∞^2
	$n = 4$	∞^2	∞^2	∞^2	∞^2

Table 7.2: Bounds for diagonal size multipartite Ramsey numbers $m_k(n, l)$ for $k, n, l = 1, \dots, 4$. The known Ramsey numbers are typeset in boldfaced characters. The bounds for the unknown numbers are inclusive and are presented in the form (upper bound : lower bound). The known Ramsey numbers and the lower bounds are motivated as follows: ¹By Proposition 2(1). ²By Proposition 2(4). ³By Proposition 2(3). ⁴By a theorem of Beineke & Schwenk, [1]. ⁵ $C_{2 \times 17}(3, 6, 7, 8, 9, 10, 11, 14)$. ⁶Due to Day *et al.*, [13]. ⁷ $C_{3 \times 8}(1, 2, 6, 7, 9, 10)$. ⁸ $C_{3 \times 12}(1, 2, 3, 7, 9, 10, 11, 12, 15)$. ⁹ $C_{4 \times 5}(2, 3, 6, 7, 9)$. ¹⁰ $C_{4 \times 8}(1, 3, 5, 6, 9, 10, 12, 14)$. The upper bounds are motivated as follows: ^aBy Theorem 6. ^bBy Proposition 1(4). ^cBy Theorem 7.